Module-1: Introduction to definition of Topological spaces

To get a proper notion of continuity in abstract case one needs a notion of nearness, so that one can say about continuous transformation.

The idea is that if one geometric object can be continuously transformed into another, then the two objects are to be viewed topologically equivalent.
For example, an open interval of length one can be stretched to an open interval of length two, a circle and a square are topologically equivalent, one can be continuously transformed into another by radial projection.
On the other hand, a closed interval is topologically distinct from a circle or square. In fact, when a point is removed from a circle it remains is still connected but if a point lying between 0 and 1 remover from a closed interval it produces two different pieces.

The term used to describe two geometric objects are topologically equivalent is homeomorphism.
Thus a circle and a square are homeomorphic. Concretely, a radial projection from a circle C to a square S with the same center point, produces a homeomorphism between C and S.
One of the basic problems of Topology is to determine when two given geometric objects are non homeomorphic, for example whether the letters M, N, H, B are non homeomorphic.
This can be quite difficult in general. Our first goal will be to define exactly what the 'geometric objects' are that one studies in Topology. These are called topological spaces.

Roughly speaking continuous function from one topological space to another is that which preserve nearness.

Beyond $\varepsilon - \delta$ definition a nice definition of a continuous function in terms of open sets:

Definition 1. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous if for each open set O in $\mathbb{R}$ the inverse image $f^{-1}(O) = \{x \in \mathbb{R} : f(x) \in O\}$ is also an open set.*

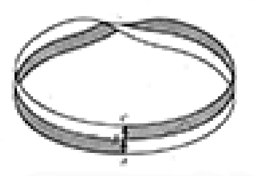
To see that this definition corresponds to the intuitive notion of continuity, let us examine, what would happen if this condition fails. There would then be an open set O for which $f^{-1}(O)$ is not open. This means there would be a point $x_0 \in f^{-1}(O)$ for which there is no interval (a, b) containing x_0 and contained in $f^{-1}(O)$.

This is equivalent to saying there would be points x arbitrarily close to x_0 those are in the complement of $f^{-1}(O)$. For x to be in the complement of $f^{-1}(O)$ means that $f(x)$ is not in O. On the other hand, x_0 was in $f^{-1}(O)$ so $f(x_0)$ is in O. Since O was assumed to be open, there is an interval (c, d) about $f(x_0)$ that is contained in O. The points $f(x)$, those are not in O are therefore not in (c, d), so they remain at least a fixed positive distance from $f(x_0)$. A reasonable interpretation of discontinuity of f at x_0 would be that there are points x arbitrarily close to x_0 for which $f(x)$ stays at least a fixed positive distance say ε away from $f(x_0)$. Let O be the open set $(f(x_0) - \varepsilon,\ f(x_0) + \varepsilon)$. Then $f^{-1}(O)$ contains x_0 but it does not contain any points x for which $f(x)$ is not in O, and we are assuming there are such points x arbitrarily close to x_0, so $f^{-1}(O)$ is not open since it does not contain all points in some interval (a, b) about x_0.

In trying to find a satisfactory definition of a topological space we shall have two aims in mind.

1. The definition should be general enough to allow a wide range of different structures as spaces. We would like to consider a finite, discrete set of points as a space, or equally a whole uncountable continuum of points such as the real line; our nice geometrical surfaces should qualify under the definition, and also sets of functions such as the set of continuous complex-valued functions defined on the unit circle in the complex plane.

2. The definition will be so that, we would be able to perform simple constructions with our spaces, such as taking the cartesian product of two spaces, or identifying some of the points of a space in order to form a new one. For example in the construction of Mobius strip we take a rectangular piece of rubber sheets and then identify two opposite sides in different direction.



The definition of a space should contain enough information so that we can define the notion of continuity for functions between spaces. It is really this second consideration which leads to the abstract definition given below.

Definition 2. *A topological space is a non empty set X together with a collection $\mathcal{O}$ of subsets of X, called open sets, such that:*

(i) Both $\emptyset$ and X are in $\mathcal{O}$.

(ii) The union of any collection of sets in $\mathcal{O}$ is in $\mathcal{O}$.

(iii) The intersection of any finite collection of sets in $\mathcal{O}$ is in $\mathcal{O}$.

The collection $\mathcal{O}$ of open sets is called a topology on X.

The most known example of topological space is $\mathbb{R}$ with usual open sets as the open sets of the topology of $\mathbb{R}$.

Example 1. *Let X be a nonempty set and $\mathcal{O} = \mathcal{P}(X)$. Then it is easy to observe that $(X, \mathcal{O})$ is a topological space called discrete space also denote by X_d.*

Example 2. *Let X be a nonempty set and $\mathcal{O} = \{\emptyset, X\}$. Then it is topological space called indiscrete space.*

Example 3. *Let $X = \{a, b, c\}$ and $\mathcal{O} = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then observed that $\mathcal{O}$ is a topology on X, so that $(X, \mathcal{O})$ is a topological space.*

Example 4. *Let X be an infinite set and let $\tau_{cf} = \{A \subset X : X \setminus A \text{ is finite}\}$. Then (X, τ_{cf}) is a topological space, usually called cofinite space.*

Proof. $\emptyset \in \tau_{cf}$ by default and $X \in \tau_{cf}$ is obvious.
Now let $A_1, A_2 \in \tau_{cf}$. Then

$$X \setminus (A_1 \cap A_2) = (X \setminus A_1) \cup (X \setminus A_2)$$

is finite as both is finite and therefore $A_1 \cap A_2 \in \tau_{cf}$.
τ_{cf} is closed under arbitrary union is obvious. □

Example 5. *Let X be an uncountable set and let $\tau_{co} = \{A \subset X : X \backslash A$ is countable or finite$\}$. Then (X, τ_{co}) is a topological space, usually called cocountable space.*

Example 6. *Let X be a plane. Let $\mathcal{O}$ consist of $\emptyset, X$ and all open disks with centre at the origin. Then clearly $\mathcal{O}$ is topology on X.*

Next we define an interesting topology. First we define the following notion.

Definition 3. *Let $X = \{0, 1, 2, ..., n, ...\} = \mathbb{N} \cup \{0\}$. For $A \subset X$, let $|A \cap [1, n]|$ denotes the cardinality of the set $A \cap [1, n]$.*

Theorem 1. *Define $\tau = \{A : 0 \notin A$ or $0 \in A$ and $\lim_{n \to \infty} \frac{|A \bigcap [1,n]|}{n} = 1\}$. Then τ is a topology on X.*

Proof. Clearly $\emptyset \in \tau$.
Observe that $\lim_{n \to \infty} \frac{|X \cap [1,n]|}{n} = 1$ so that $X \in \tau$. If $A, B \in \tau$, that is $\lim_{n \to \infty} \frac{|A \cap [1,n]|}{n} = 1$ and $\lim_{n \to \infty} \frac{|B \cap [1,n]|}{n} = 1$. Then $\lim_{n \to \infty} \frac{|A^c \cap [1,n]|}{n} = 0$ and $\lim_{n \to \infty} \frac{|B^c \cap [1,n]|}{n} = 0$. This implies that $\lim_{n \to \infty} \frac{|(A^c \cup B^c) \cap [1,n]|}{n} = 0$. Since $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$ we have the desired result. Hence τ is closed under finite intersection. That τ is closed under arbitrary union is easy.

□

As elementary real analysis we declare a set in a topological space X to be closed if it is complement of an open set.

Theorem 2. *Let X be a topological Then the following conditions hold:*

(1) $\emptyset$, X are closed sets.

(2) Arbitrary intersection of closed sets is closed

(3) Finite union of closed sets is closed.

Proof. (1) is obvious.

(2) Let $\{F_\alpha : \alpha \in \Gamma\}$ be a collection of closed sets. Then by definition $\{X \setminus F_\alpha : \alpha \in \Gamma\}$ is a collection of open sets. Using elementary set theory the result follows.

□

Like elementary real analysis we can introduce the notions of Interior, limit points, closure, boundary points for arbitrary topological spaces.

Definition 4. *Let A be a subset of a topological space X and $x \in X$. Then x is said to be a limit point of A if every open set containing x meets A at a point other that x. If x is not a limit point of A then it is an isolated point of A. Hence any isolated point of X is just an open set. A point $x \in A$ is said to be an interior point of A if there exists an open B such that $x \in B \subset A$.*

Definition 5. *Let A be a subset of a topological space X. A point $x \in A$ is said to be an interior point of A if there exists a open O in X such that $x \in O \subset A$.*

The set of all limit points of a set A is called derived set of A, By the closure of A we mean the set A along with its limit points, and denoted by $\overline{A}$. The set of all interior points of A called interior of A and denoted by A°.

From elementary analysis we already know that closure of $\mathbb{Q}$ in $\mathbb{R}$ is whole $\mathbb{R}$ and interior of $\mathbb{Q}$ is emptyset.

Question 1. *What will be the interior, closure of the following sets : (a) $\{(x,y)|1 < x^2+y^2 \leq 2\}$*

(b) $\mathbb{R}^2$ with both axes removed

Proposition 1. *Let X be a topological space, and A, B be subsets of X. Then prove the followings:*

(1) If $A \subset B$ then $\overline{A} \subset \overline{B}$,

(2)$\overline{A \cup B} = \overline{A} \cup \overline{B}$,

(3)$(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proof. (1) follows easily. For part (2) let $x \in \overline{A \cup B}$ and U be a neighborhood of x. Then $U \cap (A \cup B) \neq \emptyset$. This means that either $U \cap A \neq \emptyset$ or $U \cap B \neq \emptyset$. This implies that $x \in \overline{A} \cup \overline{B}$, so that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Conversely let $x \in \overline{A} \cup \overline{B}$ and U be a neighborhood of x. Then either $U \cap A \neq \emptyset$ or $U \cap B \neq \emptyset$, that is $U \cap (A \cup B) \neq \emptyset$ so that $x \in \overline{A \cup B}$. This gives the reverse inequality. □

Theorem 3. *Let X be a topological and $A \subset X$. Then the followings hold:*

(1) A° is the largest open set contained in A.

(2) $\overline{A}$ is the smallest closed set containing A.

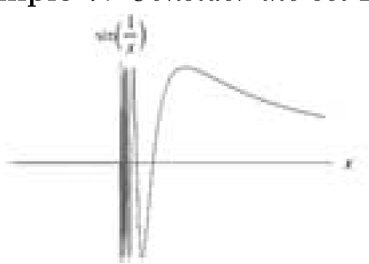
(3) A is open if and only if $A = A^\circ$.

(4) A is closed if and only if $A = \overline{A}$.

Proof. (1) If $x \in A^\circ$ then there exists some open B_x such that $x \in B_x \subset A$. We have $B_x \subset A^\circ$. In fact for each $y \in B_x$, B_x is an open set contained in A and containing y. Now let O be an open set contained in A. Then for each $x \in O$ there exists some open B_x such that $x \in B_x \subset A$, i.e. each element of O is an interior point of A so that $O \subset A^\circ$.

(2) Let $x \in X \setminus \overline{A}$. Then there exists an open set O containing x which misses $\overline{A}$. We claim that O also misses $\overline{A}$. If not so then there exists some $y \in \overline{A} \setminus A$ such that $y \in O$. This implies that O meets A, which is a contradiction. Therefore $\overline{A}$ is closed. To prove that $\overline{A}$ is the smallest closed set containing A, let K be a closed set containing A. If possible let there exists some $y \in \overline{A} \setminus K$. Then there exists an open set O containing y, which misses A, which contradicts the fact that $y \in \overline{A}$. □

Example 7. *Consider the set $\mathbb{R}^2 \setminus \{(x, sin\frac{1}{x}) : x > 0\}$.*



The graph of so called sin curve is is displayed in picture. Any point on the sin curve cann't be a limit point. So the closure of the given set is itself. Now choose a point on the set $\{0\} \times [-1, -1]$. Each point on this set is a limit point of the set $\{(x, sin\frac{1}{x}) : x > 0\}$ and not an interior point of the set in question. Hence the interior of the set in question is $\mathbb{R}^2 \setminus (\{(x, sin\frac{1}{x}) : x > 0\} \cup \{0\} \times [-1, -1])$.

We can define convergency of sequence in topological spaces analogues to metric space.

Definition 6. *Let X be a topological space and $x \in X$. A sequence $(x_n)_{n=1}^{\infty}$ is said to converge at x if for any neighborhood N_x of x there exists some $n_0 \in \mathbb{N}$ such that $(\forall n \geq n_0)(x_n \in N_x)$.*

In the following we will examine the property which is responsible for unique limit of any convergent sequence in $\mathbb{R}_u$.

Definition 7. *Let X be a topological space*

(1) X is said to be T_1 if every finite set is closed in X. (2) X is said to be Hausdorff or T_2 if any two distinct points can be strongly separated by two disjoint open sets.

Proof. Let $x \in X$ and $y \neq x$. Then there exist disjoint open sets U and V containing x and y respectively. Therefore y is not a limit point of the set $\{x\}$. This implies that every singletoned is closed and therefore every finite set is closed. □

Note that cofinite space is not Hausdorff. In fact if x and y are two distinct points U and V are two disjoint opensets containing x and y respectively, then $X \setminus U$ and $X \setminus U$ are finite. But then $(X \setminus U) \cup (X \setminus V) = X \setminus (U \cap V)$. But the left hand side being finite and the right hand side being whole X as $U \cap V = \emptyset$ we get a contradiction.

One can observe that for the cofinite space every sequence converges to every point. In this position we like to introduce the new two definitions. Here we prove that Hausdorfness is responsible for unique limit of any convergent sequence.

Proposition 2. *In a Hausdorff space any sequence can have at most one limit.*

Proof. Let $x \in X$ and $y \neq x$. Then there exist disjoint open sets U and V containing x and y respectively. Now if a sequence (x_n) converges to x then it is not possible to converge to y, as except finitely many points, all the points of the sequence are in U. □